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COMMENT

Universal $f(\alpha)$ spectrum as an eigenvalue

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Abstract. It is shown that the multifractal spectrum $f(\alpha)$ of the Feigenbaum attractor with respect to its natural measure can be computed from the largest eigenvalue of a generalised Frobenius-Perron operator. This method is easy to implement and provides very high accuracy.

In the spirit of the thermodynamic formalism [1], it has recently been shown [2-7] that scaling properties of dynamical systems can be studied via eigenvalues of certain Frobenius-Perron-type operators. For one-dimensional unimodal maps $x'=f(x)$ this eigenvalue problem can be written as a generalised Frobenius-Perron equation [3, 4, 7]

$$\lambda(\beta)Q^{(\beta)}(x') = \sum_{x\in f^{-1}(x')} \frac{Q^{(\beta)}(x)}{|f'(x)|^{\beta}}. \tag{1}$$

Here β is a real parameter $(-\infty<\beta<\infty)$, f' and f^{-1} stand for the derivative and the inverse of f, respectively. The right-hand side of (1) defines the generalised Frobenius-Perron (GFP) operator $\hat{P}(\beta)$, $\lambda(\beta)$ is an eigenvalue and $Q^{(\beta)}(x)$ is the corresponding eigenfunction. In this comment, based on recent work [6, 8, 9], we show that this equation provides a convenient way to calculate the multifractal spectrum [10] of the Feigenbaum attractor [11], which is of great importance due to its universality in the space of 1D maps with quadratic maximum.

Eigenvalues showing up in an iterative solution of (1) are of special interest. Starting from an initial function $Q_0(x)$, the iteration scheme $Q_{n+1}=\lambda^{-1}\hat{P}(\beta)Q_n$ converges for $n\to\infty$ to a non-trivial limit $Q^{(\beta)}(x)$ only if λ is a particular eigenvalue $\lambda(\beta)$ [3]. The largest and most relevant of these eigenvalues $\lambda_*(\beta)\equiv\exp(-\beta F(\beta))$ can be obtained [3] by using any smooth initial function Q_0. In [4, 7] it was shown that

$$-\ln\lambda_*(\beta)\equiv\beta F(\beta)=G(\beta) \tag{2}$$

where $G(\beta)$ is defined by the relation

$$\sum_{i=1}^{2^{n-1}} \Delta_i^{\beta} \sim \exp(-G(\beta)n) \tag{3}$$

for $n\to\infty$. The length scales, $\Delta_i = x_{2i}-x_{2i-1}$, appearing in the sum are obtained from the nth pre-images x_j $(j=1,2,3,\ldots,2^n)$ of some seed point x^*, ordered along the x axis.

The function $\beta F(\beta)$, called free energy in the thermodynamic formalism, describes purely geometric properties, namely the length scale distribution of the intervals Δ_i. However, it can contain information on the metric properties as well if the measure sitting on the intervals has a relatively simple structure [6, 9]. As a special case, if all the Δ_i are given equal measure $p_i = 1/(2^{n-1})$, the relation [10, 12]

$$\sum_{i=1}^{2^{n-1}} \frac{p_i^q}{\Delta_i^{\tau(q)}} \sim 1 \qquad \tau(q) = (q-1)D_q \tag{4}$$

defining the generalised dimensions D_q [13] can be written as

$$\sum_{i=1}^{2^n} \Delta_i^{-\tau(q)} \sim 2^{nq}. \tag{5}$$

Using (2) and (3), it immediately follows [8, 9] that

$$\beta F(\beta)|_{\beta=(1-q)D_q} = -q \ln 2. \tag{6}$$

For the multifractal spectrum $f(\alpha)$ one obtains

$$f(\alpha) = \frac{S(E)}{E}\bigg|_{E=(\ln 2)/\alpha} \tag{7}$$

where $S(E)$ is the Legendre transform of $\beta F(\beta)$:

$$S(E_\beta) = \beta E_\beta - \beta F(\beta) \qquad E_\beta = \frac{\mathrm{d}(\beta F(\beta))}{\mathrm{d}\beta}. \tag{8}$$

We will use these relations when calculating the universal $f(\alpha)$ function for the Feigenbaum attractor.

In what follows an essential role is played by the fact that the Feigenbaum attractor can be interpreted as the repeller of a certain unimodal expanding map $f_{\mathrm{E}}(x)$ [4, 14]. The attractor consists of the forward iterates $x_i = g^i(x_0)$ ($i = 0, 1, 2, 3, \ldots$) of $x_0 = g(0)$ where the map g obeys the Feigenbaum-Cvitanovic functional equation [11]: $\alpha_{\mathrm{F}} g^2(x/\alpha_{\mathrm{F}}) = g(x)$, $g(0) = 1$ with α_{F} being a universal scaling factor [11]. It was shown in [4] that the points x_i can be organised on a binary tree defined by the rule

$$x_{2j+\varepsilon} = F_\varepsilon(x_j) \qquad \varepsilon = 0, 1 \tag{9}$$

where $F_0(x) = (\alpha_{\mathrm{F}} g)^{-1}(x)$ and $F_1(x) = \alpha_{\mathrm{F}}^{-1}(x)$. The functions F_0 and F_1 (called 'presentation functions' in [4]) can then be regarded as two branches of the inverse of the following unimodal expanding map f_{E} [4, 14]:

$$f_{\mathrm{E}}(x) = \begin{cases} \alpha_{\mathrm{F}} x & x_1 \leqslant x \leqslant x_3 \\ \alpha_{\mathrm{F}} g(x) & x_2 \leqslant x \leqslant x_0. \end{cases} \tag{10}$$

It maps both regions $[x_1, x_3]$ and $[x_2, x_0]$ onto the interval $[x_1, x_0]$ also containing the 'hole' between x_3 and x_2. After a large number of iterations, the points that have not yet escaped the interval $[x_1, x_0]$ sit on a close vicinity of a Cantor set, the repeller of the map f_{E}.

It is easy to see that the points $x_0, x_1, x_2, \ldots, x_{2^n-1}$ are the nth pre-images of x_0 under the map f_{E}. Just as in the general case described in connection with the GFP equation, one can construct the length scales Δ_i from the pairs of the pre-images of the seed point $x^* = x_0$: $\Delta_i = |x_i - x_{i+2^{n-1}}|$ ($i = 0, 1, 2, \ldots, 2^{n-1} - 1$). (The formal difference between this notation and that used earlier comes from the fact that the points of the Feigenbaum attractor are indexed according to the forward iteration of g rather than their actual position on the x axis.) These intervals represent a complete covering of the repeller of f_{E}, which can be refined by increasing n.

The above considerations show that the Feigenbaum attractor can also be obtained as the repeller of the map f_E. This fact enables us to use the GFP equation applied to f_E in order to calculate its free energy. Since on the Feigenbaum attractor all the intervals Δ_i are equally probable with respect to the natural measure of the map g, the multifractal spectrum follows from (6)–(8).

To calculate $\beta F(\beta)$, the most convenient method is the iterative solution of (1). Since the largest eigenvalue $\lambda_*(\beta)$ does not depend on the particular choice of the smooth initial function $Q_0(x)$, one can use $Q_0 \equiv 1$. In this case the result of the nth step of the iteration can be given explicitly [3]:

$$Q_n(x') = \exp(n\beta F(\beta)) Z_n(\beta, x') \tag{11}$$

where

$$Z_n(\beta, x') = \sum_{x \in f_E^{-n}(x')} \frac{1}{|f_E^{(n)\prime}(x)|^\beta}. \tag{12}$$

f_E^{-n} and $f_E^{(n)\prime}$ denote the inverse and the derivative of the n-fold iterate of the map, respectively. Using the convergence of the iteration, the following scaling relation can be obtained for asymptotically large n:

$$Z_n(\beta, x') \sim \exp(-n\beta F(\beta)). \tag{13}$$

At intermediate values of n, by taking the logarithm of the ratio Z_{n-1}/Z_n, one finds

$$\ln \frac{Z_{n-1}}{Z_n} = \beta F(\beta) + d_n. \tag{14}$$

The correction d_n appears as the effect of the second-largest eigenvalue of the GFP operator. For hyperbolic repellers, it falls exponentially with increasing n: $d_n \sim (d(\beta))^n$, where the quantity $d(\beta)$, less than 1 in modulus, is the ratio of the second-largest eigenvalue to $\lambda_*(\beta)$. This type of behaviour provides fast convergence and a possibility of avoiding strong finite-size effects in the calculation.

As an input to our calculations, we used the series expansion of the function $g(x)$ up to the 14th power of x:

$$g(x) = 1 + \sum_{i=1}^{7} a_i x^{2i} + \dots$$

as given in [11]. The value of x' in (11) was fixed to be 0.1 (the free energy should not depend on x') and its pre-images were determined up to the 12th generation. $\beta F(\beta)$ was deduced from the n-dependence of the quantity Z_n by using (13) and (14). The exponential decay of d_n was also verified in this case. This fact enabled us to calculate the free energy with very high accuracy by estimating the corrections due to the finite value of n as the sum of an infinite geometric series of quotient $d(\beta)$. (The modulus of $d(\beta)$ was found for all β values to be less than 0.5.) The free-energy curve computed in this way is shown in figure 1. The convergence was very fast: six-digit precision was reached even after the $n=6$ iteration, while for the $n=12$ step we obtained an accuracy of less than 10^{-10}.

The generalised dimensions D_q and the multifractal spectrum $f(\alpha)$ with respect to the natural measure on the Feigenbaum attractor was determined by using (6)–(8). The high accuracy of these quantities follows from that of the function $\beta F(\beta)$. Our results for D_q and $f(\alpha)$ were found to be in complete agreement with former results published in the literature [10, 15]. As a particular example, we obtained $D_0 = 0.538$

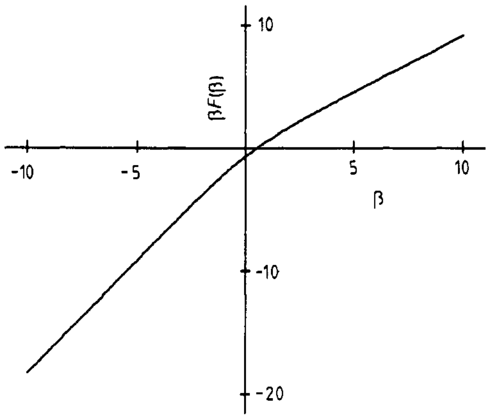

Figure 1. The free energy $\beta F(\beta)$ of the Feigenbaum attractor in the range $-10 < \beta < 10$, obtained by the iterative method after the $n = 12$ iteration.

045 143 5(3) for the Hausdorff dimension via the relation $F(D_0) = 0$. This value is consistent with the results of [12, 15, 16] and agrees up to an accuracy of 10^{-10} with the most precise of them [15]. Table 1 shows a few values from the D_q spectrum obtained by the above method with a ten-digit accuracy. The $f(\alpha)$ curve is indistinguishable from the results published earlier [10, 15], so it is not exhibited here.

It is worth noting that our method can be used to calculate, besides the multifractal spectrum, other quantities which are also universal characteristics of quadratic maps at the onset of chaos. The quantity $d(\beta)$ describing the speed of convergence is the ratio of two eigenvalues of the GFP operator of the universal map f_E, so its universality is obvious. Recently, another important universal function A_q was introduced in [17] to describe finite-size effects in the calculation of the generalised dimensions based on the thermodynamic formalism. A detailed comparison shows that this quantity is connected with the limit function $Q^{(\beta)}(x)$ of the iteration procedure taken at $\beta = (1-q)D_q$.

In this comment we showed that the universal $f(\alpha)$ spectrum can be determined with high precision from an eigenvalue problem. Our results show clearly that the

Table 1. Generalised dimensions D_q of the Feigenbaum attractor with respect to its natural measure for integer q values between -5 and 5, obtained by determining $\beta F(\beta)$ via the iterative method after the $n = 12$ iteration and by using (6).

q	D_q
−5	0.637 605 1836
−4	0.621 265 9425
−3	0.602 478 1718
−2	0.581 736 0003
−1	0.559 912 9101
0	0.538 045 1435
1	0.517 097 5725
2	0.497 836 4592
3	0.480 776 8494
4	0.466 151 5569
5	0.453 922 7023

method based on the GFP equation and recent advances on the thermodynamic formalism provides the maximum accuracy allowed by the incomplete knowledge of the Taylor series of $g(x)$. The precision is limited mainly by the finite accuracy (ten digits) of the Taylor coefficients. Another (and less important) source of uncertainty is the cut-off at the 14th power in the expansion.

Compared with other high-precision calculations using the same series [15], the main advantage of our method lies in its conceptual simplicity and the ease of implementation: programmed in FORTRAN, it takes less than 100 lines. The applicability of the procedure we presented is not restricted to the Feigenbaum attractor: it can be applied to other universal problems as well where presentation functions are known and the natural measure has a simple structure [4].

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